Basit kesirlere ayırma yöntemi

$f(x)=\frac{P(x)}{Q(x)}$ şeklinde rasyonel bir fonksiyon olsun.

der $P(x)=n$, der $Q(x)=m$ sayılarına göre inceleme yapalım.

1) $n \geq m$ ise polinom bölmesi yapılır.

$$\begin{array}{r|l} P(x) & Q(x) \\ \cline{2-2} & B(x) \\ \hline K(x) & \end{array} \Rightarrow \int \frac{P(x)}{Q(x)}\,dx = \int B(x)\,dx + \int \frac{K(x)}{Q(x)}\,dx$$

2) $n < m$ ve $Q(x)=(x-a_1)(x-a_2)\cdots(x-a_m)$ şeklinde farklı $a_1, a_2, \ldots, a_m$ köklerine sahip ise

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_m}{x-a_m}$$

şeklinde yazılıp payda eşitlenerek A_i ler bulunur.

Örnek: $\int \frac{x^3+2x^2}{x^2+1}\,dx = ?$

$$\begin{array}{r|l} x^3+2x^2 & x^2+1 \\ \cline{2-2} \underline{-x^3+x} & x+2 \\ 2x^2-x & \\ \underline{-2x^2+2} & \\ -x-2 & \end{array}$$

$$\Rightarrow \frac{x^3+2x^2}{x^2+1} = x+2-\frac{x+2}{x^2+1}$$

$$\Rightarrow \int \frac{x^3+2x^2}{x^2+1}\,dx = \int\left(x+2-\frac{x+2}{x^2+1}\right)dx = \int (x+2)\,dx - \int \frac{x+2}{x^2+1}\,dx$$

$$= x^2+2x - \int \frac{x}{x^2+1}\,dx - 2\int \frac{1}{x^2+1}\,dx$$

$$= x^2+2x - \frac{\ln(x^2+1)}{2} - 2\arctan x + c.$$

Örnek: $\int \frac{x-5}{x^2-5x+6}\,dx = ?$

$$\frac{x-5}{x^2-5x+6} = \frac{x-5}{(x-3)(x-2)} = \frac{A}{\underset{(x-2)}{x-3}} + \frac{B}{\underset{(x-3)}{x-2}} = \frac{Ax-2A+Bx-3B}{x^2-5x+6}$$

$\Rightarrow \quad x-5=(A+B)x-2A-3A \quad \Rightarrow \quad A+B=1 \;,\; -2A-3B=-5$

$$\Rightarrow \begin{array}{l} 3/\; A+B=1 \\ \;\; -2A-3B=-5 \\ \hline \;\; A=-2 \;,\; B=3 \end{array} \qquad \Rightarrow \quad \frac{x-5}{x^2-5x+6}=\frac{3}{x-2}-\frac{2}{x-3}$$

$$\Rightarrow \int \frac{x-5}{x^2-5x+6}\,dx=\int\frac{3}{x-2}\,dx-\int\frac{2}{x-3}\,dx=3\ln|x-2|-2\ln|x-3|+c$$

3) $Q(x)$ in katlı kökleri varsa

$$\frac{P(x)}{Q(x)}=\frac{P(x)}{(x-a_1)^k(x-a_2)^r(x-a_3)\dots(x-a_p)}$$

$$=\frac{A_1}{x-a_1}+\frac{A_2}{(x-a_1)^2}+\dots+\frac{A_k}{(x-a_1)^k}+\frac{B_1}{x-a_2}+\frac{B_2}{(x-a_2)^2}+\dots+\frac{B_r}{(x-a_2)^r}$$

$$+\frac{C_1}{x-a_3}+\frac{C_2}{x-a_4}+\dots+\frac{C_{p-2}}{x-a_p}$$

yazılarak A_i, B_j, C_k sabitleri bulunup işlem yapılır.

Örnek: $\displaystyle\int \frac{x^2-2}{(x-1)^3}\,dx = ?$

$$\frac{x^2-2}{(x-1)^3} = \underset{(x-1)^2}{\frac{A}{x-1}} + \underset{(x-1)}{\frac{B}{(x-1)^2}} + \frac{C}{(x-1)^3} = \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3}$$

$$\Rightarrow \quad x^2-2 = A(x-1)^2 + B(x-1) + C = Ax^2 - 2Ax + A + Bx - B + C$$
$$= Ax^2 + (B-2A)x + A - B + C$$

$$\Rightarrow \quad A = 1, \quad \underbrace{B-2A=0}_{B=2A=2}, \quad \underbrace{A-B+C=-2}_{1-\cancel{2}+C=-\cancel{2} \;\Rightarrow\; C=-1}$$

$$\Rightarrow \quad \frac{x^2-2}{(x-1)^3} = \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{(x-1)^3}$$

$$\Rightarrow \int \frac{x^2-2}{(x-1)^3}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{2}{(x-1)^2}\,dx - \int \frac{1}{(x-1)^3}\,dx$$

$$= \ln|x-1| - \frac{2}{x-1} + \frac{1}{2(x-1)^2} + c$$

4) $Q(x)=0$ denkleminin 2. dereceden çarpanı varsa yani $Q(x)=(x-a_1)(x-a_2)\ldots(cx^2+dx+e)^k$ şeklinde ise

$$\frac{P(x)}{Q(x)}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\cdots+\frac{C_1x+D_1}{cx^2+dx+e}+\frac{C_2x+D_2}{(cx^2+dx+e)^2}+\cdots+\frac{C_kx+D_k}{(cx^2+dx+e)^k}$$

yazılarak A_i, C_j, D_t bulunur.

Örnek: $\displaystyle\int \frac{x^2+3x+2}{x^3+x}\,dx=?$

$$\frac{x^2+3x+2}{x^3+x}=\frac{x^2+3x+2}{x(x^2+1)}=\underset{(x^2+1)}{\frac{A}{x}}+\underset{(x)}{\frac{Bx+C}{x^2+1}}=\frac{Ax^2+A+Bx^2+Cx}{x^3+x}$$

$$\Rightarrow x^2+3x+2=(A+B)x^2+Cx+A \quad\Rightarrow A=2,\ B=-1,\ C=3$$

$$\Rightarrow \int\frac{x^2+3x+2}{x^3+x}\,dx=\int\frac{2}{x}\,dx+\int\frac{3-x}{x^2+1}\,dx=2\int\frac{1}{x}\,dx+3\int\frac{1}{x^2+1}\,dx-\int\frac{x}{x^2+1}\,dx$$

$$=2\ln|x|+3\arctan x-\frac{1}{2}\ln(x^2+1)+c.$$

Örnek: $I=\int \frac{2x-3}{x^3+x} = ?$

$$\frac{2x-3}{x^3+x} = \frac{2x-3}{x(x^2+1)} = \underset{(x^2+1)}{\frac{A}{x}} + \underset{(x)}{\frac{Bx+C}{x^2+1}} = \frac{Ax^2+A+Bx^2+Cx}{x^3+x} = \frac{(A+B)x^2+Cx+A}{x^3+x}$$

$\Rightarrow A+B=0,\ C=2,\ A=-3 \Rightarrow B=3$

$$\Rightarrow \frac{2x-3}{x^3+x} = -\frac{3}{x} + \frac{3x+2}{x^2+1}$$

$$\Rightarrow I = -3\underbrace{\int \frac{dx}{x}}_{\ln|x|} + \int \frac{3x+2}{x^2+1}\,dx = -3\ln|x| + 3\int \frac{x}{x^2+1}\,dx + 2\int \frac{dx}{x^2+1}$$

$$= -3\ln|x| + \frac{3}{2}\ln(x^2+1) + 2\arctan x + c$$

Örnek: $\int \frac{dx}{e^{2x}-4e^{x}+4} = ?$

$$e^{x} = u \Rightarrow e^{x}dx = du \Rightarrow dx = \frac{du}{e^{x}} = \frac{du}{u}$$

$$\int \frac{du}{u(u^2-4u+4)} = \int \frac{du}{u(u-2)^2}$$

$$\frac{1}{u(u-2)^2} = \frac{A}{\underset{(u-2)^2}{u}} + \frac{B}{\underset{u(u-2)}{u-2}} + \frac{C}{\underset{u}{(u-2)^2}} = \frac{A(u-2)^2 + Bu(u-2) + Cu}{u(u-2)^2}$$

$$\Rightarrow 1 = A(u-2)^2 + Bu(u-2) + Cu$$

$u=0$ için $1 = A(-2)^2 = 4A \Rightarrow A = \frac{1}{4}$

$u=2$ için $1 = 2C \Rightarrow C = \frac{1}{2}$

$\Rightarrow 1 = \frac{1}{4}(u-2)^2 + Bu(u-2) + \frac{u}{2} \Rightarrow$ işlemler yapılırsa $B = -\frac{1}{4}$ olarak bulunur.

$$\Rightarrow \frac{1}{u(u-2)^2} = \frac{1}{4u} - \frac{1}{4(u-2)} + \frac{1}{2(u-2)^2}$$

$$\Rightarrow \int \frac{du}{u(u-2)^2} = \frac{1}{4}\int \frac{du}{u} - \frac{1}{4}\int \frac{du}{u-2} + \frac{1}{2}\int \frac{du}{(u-2)^2}$$

$$= \frac{1}{4}\ln|u| - \frac{1}{4}\ln|u-2) - \frac{1}{2(u-2)} + c$$

$$(e^x = u \text{ idi}) \quad = \frac{x}{4} - \frac{1}{4}\ln(e^x-2) - \frac{1}{2(e^x-2)} + c$$